HW5 — 4050 (you may defer Q1 to the end of Ch.3)

1. Let $m(E) \leq +\infty$, $\mathcal{M}_F^+(E) \ni f : E \to [0,\infty]$, measurable.

For each $n \in \mathbb{N}$, let $E_n := E \cap [-n, n]$

$A_{n,k} := \{ x \in E_n : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \}$, $k = 1, 2, \cdots, 2^n, \cdots 2 \cdot 2^n, \cdots 3 \cdot 2^n, \cdots, n2^n$.

$B_n := \{ x \in E_n : n \leq f(x) \}$ (all in $\mathcal{M}$ & $E = (\dot\bigcup_{k=1}^{n2^n} A_{n,k}) \dot\cup B_n$)

and let $\varphi_n := n\chi_{B_n} + \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \chi_{A_{n,k}}$, i.e. $\forall x \in E \cap [-n, n]$,

$$\varphi_n(x) = \begin{cases} n & \text{if } n \leq f(x) \\ \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \text{ with } k = 1, 2, \cdots, n \cdot 2^n \end{cases}$$

Show that $\mathcal{S}_0^+(E) \ni \varphi_n \uparrow f$ (pointwisely on E). This result may also be referred to as Littlewood's 2nd Principle.

2. Let $F = \dot\bigcup_{n=1}^{N} F_n$, disjoint closed sets $F_1, \cdots, F_n$.

Let $f : F \to \mathbb{R}$ be such that $f|_{F_n}$ is cts, $\forall n$.

Show that f is cts.

3*. Let $F_n \subseteq (n, n+1]$ be closed ($\mathbb{R} \setminus F_n$ open) $\forall n \in \mathbb{N}$, and let $F = \dot\bigcup_{n \in \mathbb{N}} F_n$.

Show that $f : F \to \mathbb{R}$ is continuous if each $f|_{F_n}$ is cts. (The condition $F_n \subseteq (n, n+1]$ cannot be dropped, e.g. F_n is a singleton, F is countable — closed or not for F)

4. Let $\emptyset \neq F \subseteq \mathbb{R}$ be closed and $f: F \to \mathbb{R}$ be cts. Show (the Tieze Extension Th): f can be continuously extended to be on the whole of $\mathbb{R}$, via the following elementary method: let $G := \mathbb{R} \setminus F$ ($\neq \emptyset$, wlg) $= \bigcup_{n=1}^{\infty} {}_{\circ} I_n$ disjoint open intervals, by ⋯.

Then $\bar{I}_n \setminus I_n \subseteq F \ \forall n$ ($\bar{I}_n$ is the closed interval, the closure of I_n), and f can be extended to $\bar{f}$ so as $\bar{f}|_F = f$, & $\bar{f}$ is "linear" on each $\bar{I}_n$ (& cts); hence $\bar{f}$ is cts: $\forall x_0 \in \mathbb{R}$, $\bar{f}$ is cts at x_0, that is $\bar{f}$ is right-cts (& left-cts, similarly)

For the right-cts of x_0 we assume w.l.g (trivially true otherwise) that $x_0 \in F$.

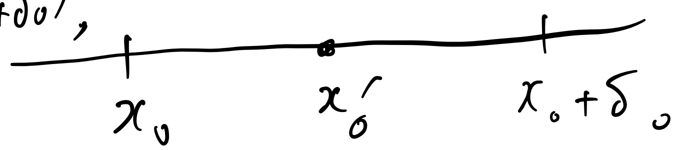
and that $\forall \delta > 0$,

$V_\delta^+(x_0) = (x_0, x_0+\delta)$ intersects both F and G. (#)

Let $\varepsilon > 0$. By the given continuity of f on F $\exists\, \delta_0 > 0$ s.t.

(*) $|f(x) - f(x_0)| < \varepsilon \quad \forall\, x \in F \cap [x_0 - \delta_0, x_0 + \delta_0]$

By (#), we take $x_0' \in F \cap (x_0, x_0+\delta_0)$,



and it follows that

(**) $|\bar{f}(x) - \bar{f}(x_0)| < \varepsilon \quad \forall\, x \in (x_0, x_0')$ (so "right-cts")

Since this already true if $x \in F$, we only need to consider the case $x \in G \cap (x_0, x_0')$ with $x \in I_n$ for some n. Now, as $x_0, x_0' \in F$, this entails that $x \in I_n \subseteq (x_0, x_0')$

The end-pts ($\bar{x}$ say) of I_n are not in G so $\bar{x} \in F$ and it follows from (*) that

$$\left| f(\bar{x}) - f(x_0) \right| < \varepsilon$$

$$\shortparallel$$

$$\left| \bar{f}(\bar{x}) - \bar{f}(x_0) \right| < \varepsilon$$

and, by the def of $\bar{f}$ on $\bar{I}_n$, it follows that

$$\left| \bar{f}(\cdot) - \bar{f}(x_0) \right| < \varepsilon \text{ on } I_n \ni x$$

so (**) is check for $x \in G \cap (x_0, x_0')$, and hence $\forall\, x \in (F \cup G) \cap (x_0, x_0') = (x_0, x_0')$, the whole interval.

5.* Check (similarly to Q4) the left-continuity of $\bar{f}$ at x_0.